Absorbing phase transition in a conserved lattice gas model in one dimension

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(Received 26 June 2007; revised manuscript received 14 November 2007; published 14 February 2008)

We investigated the nonequilibrium phase transition of the conserved lattice gas model in one dimension using two update methods: i.e., the sequential update and the parallel update. We measured the critical indices of θ, β, $\nu_{\|}$, and $\nu_{\perp}$ and found that, for a parallel update, the exponents were delicately influenced by the hopping rule of active particles. When the hopping rule was designed to be symmetric, the results were found to be consistent with those of the sequential update. The exponents we obtained were precisely the same as the corresponding results of a recently presented lattice model of two species of particles with a conserved field in one dimension, in contrast with the authors' claim. We also found that one of the scaling relations known for absorbing phase transition is violated.

DOI: 10.1103/PhysRevE.77.021113 PACS number(s): 05.70.Ln, 05.50.+q, 64.60.Ht

The nonequilibrium phase transition from a fluctuating phase into one or several absorbing states has attracted considerable attention in the past decades [1–4]. In the nonequilibrium phase transition, absorbing phase transitions with stochastic dynamics can be ubiquitously observed, particularly in condensed matter physics and population modelings [1]. The most prominent and robust universality class is the directed percolation (DP) class [5]. A wide variety of models with quite different evolution rules were found to belong to the DP class [1–4], supporting the DP hypothesis presented by Janssen [6] and Grassberger [7]. There are other known universality classes such as the parity-conserving (PC) class [8–11] and the pair-contact process with diffusion (PCPD) class [12–14], though the latter is still controversial [15–17].

Recently, it has been discovered that the absorbing phase transition with order parameters locally coupled to a nondiffusive conserved field generated a new universality class [18] which characterizes a vast set of systems including the stochastic sandpile model [19–21], the conserved threshold transfer process (CTTP) [22], and the conserved lattice gas (CLG) model [18]. The CLG model, proposed by Rossi *et al.* [18], has a stochastic short-range interaction and exhibits a continuous phase transition to an absorbing state at a critical density of particles. A particle is defined as being active if it has at least one occupied nearest-neighbor site and, otherwise, it is considered to be inactive. The dynamics proceeds by a repulsive interaction; i.e., each active particle tends to hop to one of its nearest-neighbor empty sites. Thus, there is no particle creation or annihilation, no external fields, and no self-diffusion; therefore, the number of particles is conserved during the dynamic process. If the initial density of particles, ρ, is small, the density of active particles, ρ_a, will decrease rapidly and the system will go into an absorbing state. On the other hand, if ρ is sufficiently large, ρ_a will saturate to a steady-state value of ρ_{sat}, which is considered to be an order parameter. Thus, at the critical density $\rho=\rho_c$, ρ_a shows a power-law decay,

$$\rho_a \sim t^{-\theta}, \tag{1}$$

θ being the universal critical exponent. Above ρ_c, the order parameter shows a power law against the distance from the criticality—i.e.,

$$\rho_{\mathrm{sat}} \sim (\rho - \rho_c)^{\beta}. \tag{2}$$

Lübeck calculated, by extensive Monte Carlo simulations of two, three, four, and five dimensions, the critical indices of the order parameter and the order parameter fluctuation. It was found that the upper critical dimension of the CLG model is 4 [23].

In one dimension, since the dynamic process of the CLG model becomes deterministic due to a dimensional reduction, the critical exponents may be calculated analytically. Indeed, an analytical solution for the one-dimensional CLG model was presented by de Oliveira [24] and exact values of the critical exponents $\beta=\nu_{\perp}=1$ were obtained [25], $\nu_{\perp}$ being the exponents describing the spatial correlation length near the critical point. Park *et al.* designed a one-dimensional model of two species of particles having two symmetric absorbing states with a conserved field and obtained various exponents [26]. Based on their results, they claimed that their model displayed different critical exponents from the known models, including the ordinary CLG model.

The two update methods—i.e., the parallel update and the sequential update—are alternatively employed in the absorbing phase transition with the belief that the two update methods would yield the same critical behavior; however, we found that the critical exponents are sensitively influenced by the update rules for the CLG model in one dimension and the results from the two update methods were consistent only when the hopping rule of active particles in the parallel update was elaborately designed to be symmetric. In this paper, we present the results of the two update methods, together with the yet unknown exponents of the CLG model. We found that the critical exponents of the CLG model were consistent with those of the model by Park *et al.* [26] in one dimension. We also found that at least one of the scaling relations among the critical exponents known for the absorbing phase transition is violated.

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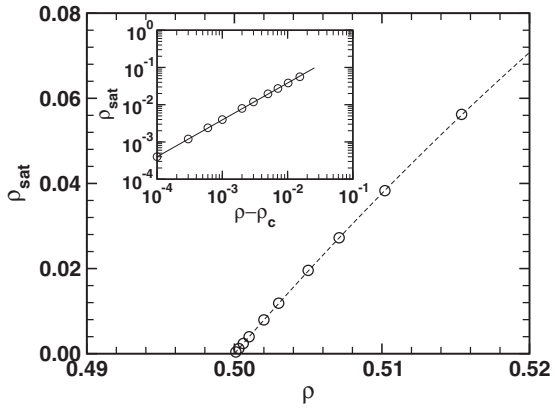

FIG. 1. The order parameter ρ_{sat} as a function of the density ρ of the particles obtained using the sequential update. The symbols are the Monte Carlo data and the dotted line is the analytical result presented in Ref. [24]. The inset shows the same data plotted on a double-logarithmic scale as a function of the distance from the criticality. The solid line represents the power-law fit with $\beta=0.984$.

The CLG model in one dimension is simple; the absorbing phase is one of the two symmetric states $010101\cdots$ and $101010\cdots$ at the critical density of $\rho_c=0.5$. In a random sequential update, initially ρL particles were distributed randomly in a system of size L. For each time step, an active particle was selected randomly and hops to the nearest-neighbor empty site, with the evolution time increasing by $\Delta t=1/N_a$, N_a being the number of active particles.

Our Monte Carlo data exhibited that ρ_a decayed for $\rho<0.5$ and saturated for $\rho>0.5$; the exponent estimated from the regression fit at $\rho_c=0.5$ was $\theta=0.246(4)$ (not shown). The order parameter ρ_{sat} was estimated from the data for $\rho>0.5$ and is plotted in Fig. 1 against the distance from the criticality. The symbols are our data and the dotted line is the analytical result by de Oliveira [24]. The inset is the plot on a double-logarithmic scale. (It should be noted that we generated up to 10^9 time steps to measure the order parameters because of extremely slow convergence behavior.) In the main plot, the data show a slight curvature, implying that the exponent β might be slightly away from the known result of $\beta=1.0$ or there might be a correction term; the regression fit in the inset shows $\beta=0.984(9)$. If, however, only data close to ρ_c were used in the regression fit, the value of β would have been closer to 1.0; therefore, our result is consistent with the exact result of $\beta=1$.

The active particle density in the vicinity of the critical point is a function of two variables t and $\xi_\parallel$, where the temporal correlation length $\xi_\parallel$ is described by the distance from criticality as $\xi_\parallel \sim |\rho-\rho_c|^{-\nu_\parallel}$. The scaling concept in the usual critical phenomena allows us to write ρ_a as a function of a single variable $t/\xi_\parallel$—i.e.,

$$\rho_a = t^{-\theta} g(t|\rho-\rho_c|^{\nu_\parallel}), \tag{3}$$

where $g(x)$ is the off-critical scaling function. Since $\rho_a \to \rho_{\mathrm{sat}} \sim |\rho-\rho_c|^{\beta}$ in the limit of $t\to\infty$ and above ρ_c, the scaling relation

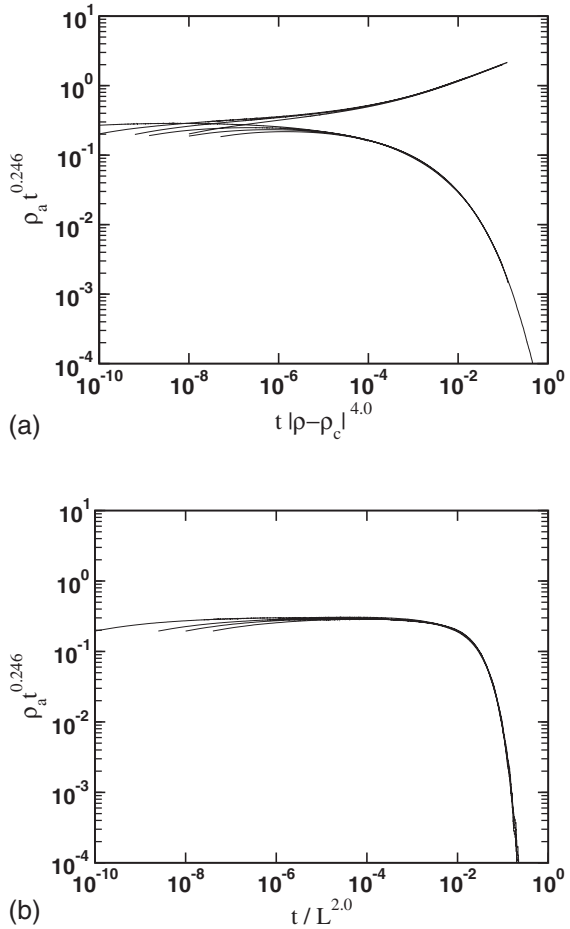

FIG. 2. Scaling functions of the active particle density obtained using the sequential update. (a) is the off-critical scaling function $\rho_a t^\theta$ plotted against the scaled variable $t|\rho-\rho_c|^{\nu_\parallel}$ for $\rho=0.515$, 0.51, 0.505, and 0.503 (upper set) and for $\rho=0.485$, 0.490, 0.494, and 0.497 (lower set), all for system size $L=10^6$. (b) is the finite-size scaling function $\rho_a t^\theta$ plotted against the scaled variable t/L^z for $L=2\times10^3$, 5×10^3, 10^4, 2×10^4, and 10^5.

$$\beta = \theta\nu_\parallel \tag{4}$$

follows. For finite-size systems, on the other hand, since the spatial correlation length is of the order of the size of system—i.e., $\xi_\perp \sim L$—it is obtained that $|\rho-\rho_c| \sim \xi_\perp^{-1/\nu_\perp} \sim L^{-1/\nu_\perp}$. Therefore, Eq. (3) can be written as a finite-size scaling form

$$\rho_a = t^{-\theta} h(t/L^z), \tag{5}$$

where z is the dynamic exponent and is known to be

$$z = \nu_\parallel/\nu_\perp. \tag{6}$$

In the simulations, since all samples for $\rho>\rho_c$ survive and exhibit steady-state densities and those for $\rho<\rho_c$ fall into one of the absorbing states if the size of system is sufficiently large and unless ρ is too close to ρ_c, ρ_a should be obtained over *all samples*. The scaling function $g(x)$ against the scaled variable $x\equiv t|\rho-\rho_c|^{\nu_\parallel}$ with the value of $\nu_\parallel$ obtained from Eq. (4)—i.e., with $\nu_\parallel=4.0$—is plotted in Fig. 2(a). Data with dif-

ferent particle densities collapse onto two separate curves, one for $\rho > \rho_c$ (upper curve) and the other for $\rho < \rho_c$ (lower curve). This assures us that our estimates of the critical indices are accurate and the scaling relation in Eq. (4) is correct. We also plotted the scaling function in Eq. (5) for various size systems, assuming the dynamic exponent z as a parameter. The best collapsing was observed for $z=2.0$, as shown in Fig. 2(b).

The value of $\nu_\perp$ can be obtained from the finite-size scaling of the steady-state densities which may be written as a function of the size of system L and the spatial correlation length $\xi_\perp$, but not a function of separate variables but of a ratio of two—i.e.,

$$\rho_{\rm sat} = |\rho - \rho_c|^\beta f_1(L|\rho - \rho_c|^{\nu_\perp}) = L^{-\beta/\nu_\perp} f(L|\rho - \rho_c|^{\nu_\perp}). \quad (7)$$

At $\rho=\rho_c$, the steady-state density scales as $\rho_{\rm sat} \sim L^{-\beta/\nu_\perp}$. In this scaling, $\rho_{\rm sat}$ should be obtained from those samples which survived and exhibited the steady-state densities. We found that, in the steady state, the only remaining active particles are those in a single dimer for all sizes—i.e., $\rho_{\rm sat} = 2/L$ yielding $\beta/\nu_\perp = 1$ or $\nu_\perp = 1$, which is identical to the exact result by de Oliveira [25].

Summarizing, the results of the random sequential update and presenting them in a compact form, we suggest

$$\theta = \frac{1}{4}, \quad \beta = 1, \quad \nu_\parallel = 4, \quad \nu_\perp = 1, \quad z = 2.$$

Surprisingly, the scaling relation in Eq. (6) does not hold with these values. A similar violation of the scaling relation was previously found by Rossi *et al.* [18]. They found that, with the relation in Eq. (4), simple scaling behavior was broken with their data. They, instead, obtained $\nu_\parallel$ from the relation in Eq. (6) using the value of z estimated from the finite-size scaling plot of Eq. (5). If we do similarly, we would get $\nu_\parallel = 2$; however, this value does not yield the data collapsing.

The scaling relation in Eq. (6) may also be obtained as follows. At the critical point, the spatial and temporal correlation lengths are given, respectively, as $\xi_\perp \sim |\rho - \rho_c|^{-\nu_\perp}$ and $\xi_\parallel \sim |\rho - \rho_c|^{-\nu_\parallel}$, and they are related to each other as $\xi_\perp \sim \xi_\parallel^{\nu_\perp/\nu_\parallel}$. At ρ_c, the spatial correlation length should scale in the same manner as the rms spreading distance R—i.e., $\xi_\perp \sim R \sim t^{1/z}$—and the temporal correlation length scales as $\xi_\parallel \sim t$, yielding the scaling relation in Eq. (6). The value of z may be obtained by studying the evolution of activity in systems close to an absorbing state. From the configuration of particles in every other site of the system, a particle near the center is moved to one of its neighbor empty sites. Then, the perturbed particle and the particle on its neighboring site become active and the system evolves in time until the perturbed particle comes back to its original site. The pair of active particles follows a random walk and the spreading distance grows in time as $R \sim t^{1/2}$, yielding $z=2$, which is identical to that we obtained from the finite-size scaling in Eq. (7). We however believe that the relation $\xi_\perp \sim \xi_\parallel^{\nu_\perp/\nu_\parallel}$ is valid only when the system is free from the finite-size effect. If the system suffers from the finite-size effect, the spatial correlation length will be of the order of the size of system—i.e., $\xi_\perp \sim L$. In the numerical simulation, it is clear that systems of the CLG model suffer from a strong finite-size effect. At ρ_c, we were not able to observe a full power-law behavior of the active particle density; we were always faced with a sharp decrease apparently caused by a finite-size effect. Such an anomalous finite-size effect was caused because the dynamics begins with particles occupied randomly and the boundary of a system immediately affects the dynamics.

We understand that the "simple scaling behavior" which Rossi *et al.* claimed to be broken is the off-critical scaling in Eq. (3). In order to examine numerically in two dimensions, we calculated the average $\rho_a(t)$ over all samples and also over surviving samples on a square lattice using the random sequential update rule. We obtained $\theta \simeq 0.412$, $\beta \simeq 0.638$, and $\beta/\nu_\perp \simeq 0.80$, which are consistent with the known corresponding values [18,27]. (However, the critical density was found to be $\rho_c = 0.3471$, which is different from that of Rossi *et al.* and also from that of Lübeck [23]. The difference might be attributed to the different averaging procedure.) We found that the scaling form in Eq. (3) was satisfied with our data on a square lattice and also with those of Lee and Kim on a Sierpinski gasket [28] as well as those in one dimension [Fig. 2(a)]. (Results on a square lattice will be published elsewhere.) We therefore were not able to find any evidence that the relation in Eq. (4) breaks the simple scaling behavior. However, it should be noted that the averaging procedure of our simulation is different from that of Rossi *et al.* Whereas in Ref. [18], only the *surviving samples* were averaged at ρ_c and the anomalous behavior of decreasing and saturating density was observed, the *all-sample* averages were calculated in our work and no such anomalous behavior was seen but the density of active particles decayed more rapidly than exponential.

Recently, Lübeck and Misra calculated the value of $\nu_\parallel$ using the persistence distributions on a square lattice [29] and derived a similar conclusion to that of Rossi *et al.* We employed the same method in one dimension. The persistence distribution—i.e., the distribution of average time that the system persists in the same phase (e.g., in a phase of $\rho_a > \langle \rho_a \rangle$)—is known to scale near the criticality as

$$P(t) = t^{-\theta_g} \mathcal{P}_1(t|\rho - \rho_c|^{\nu_\parallel}) = |\rho - \rho_c|^{\nu_\parallel \theta_g} \mathcal{P}(t|\rho - \rho_c|^{\nu_\parallel}), \quad (8)$$

where θ_g is the global persistence exponent. Our scaled data of the persistence distribution in one dimension are shown in Fig. 3 against the scaled variable. We found that the data for different values of ρ exhibited good collapsing with the adjusted parameters $\nu_\parallel = 4.0$ and $\theta_g = 1.5$. This again supports our earlier assertion that the scaling relation in Eq. (4) is valid.

The critical exponents obtained from sequential update are precisely the same as those of Park *et al.* [26] for the model of two species having two symmetric absorbing states with a conserved field. It is thus clear that the model by Park *et al.* exhibits the same critical behavior as for the CLG model in one dimension. In the lattice model by Park *et al.*, each lattice site was filled with either an A particle or a B particle and the particles with the same species in one of the nearest-neighbor sites were considered to be active. Each active particle may be exchanged with one of the neighbor-

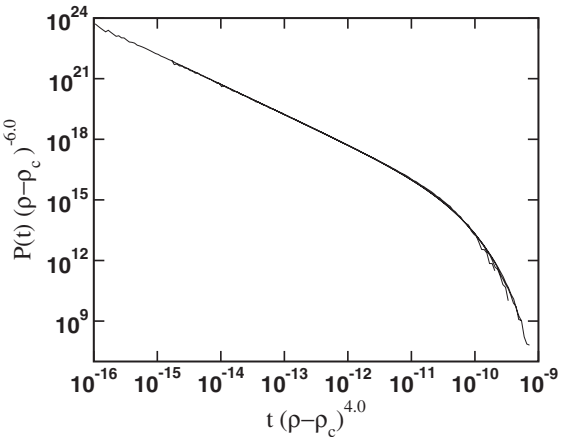

FIG. 3. The scaled persistence distribution $P(t)|\rho-\rho_c|^{\nu_\parallel\theta_g}$ in a phase of $\rho_a>\langle\rho_a\rangle$ plotted against the scaled variable $|\rho-\rho_c|^{\nu_\parallel}t$, with the values of $\theta_g=1.5$ and $\nu_\parallel=4.0$. Data are obtained for $L=40000$ and $\rho=0.5001$, 5002, 0.5003, and 0.5004.

ing particles of a different species. The parameter governing the critical behavior is the density of A particles—i.e., $p=\rho_A$—and the order parameter is the stationary density of the pairs of A particles. It is clear that the absorbing state at criticality (i.e., at $p_c=0.5$) is one of the two absorbing states $ABABAB\cdots$ and $BABABA\cdots$. They claimed that this Z_2 symmetry at criticality yielded new results; however, as we mentioned earlier, such symmetric absorbing states also existed in the original CLG model. Considering that empty sites in the CLG model correspond to the sites filled with B particles, the model by Park *et al.* would be identical to the CLG model if one imposes an additional exchange rule that an empty site (B particle) is active if it has at least one nearest-neighbor empty site (B particle) and such an active site may be filled with a particle in the neighboring site (exchange with an A particle), leaving the neighboring site empty. Our results indicated that such an additional exchange rule is irrelevant in one dimension. In higher dimensions, however, it causes the isolated A particle to diffuse and the density of active particles becomes nozero for any $\rho>0$. Therefore, the model does not undergo a phase transition, nor has the Z_2 symmetry on which Park *et al.* intended to focus.

In the parallel-update simulations, all active particles are searched and new positions to which active particles are going to hop in the next time step are determined. In this procedure, there are subtleties in determining the hopping of an active particle when two particles attempt to hop to the same site and also when two active particles that consist of a dimer attempt to hop. We present the results obtained by different rules for these cases.

In the first case, both particles in a dimer are allowed to repel and simultaneously hop to the nearest-neighbor empty sites by a repulsive interaction. In this rule, we find that there are two states at ρ_c: one is the stable absorbing state $010101\cdots$ and the other is the oscillating state having pairs of dimers with two empty sites in between ($\cdots100110011001\cdots$), but we found that the latter state is the dominant one. Particles in a dimer repel each other and consist of new dimers with repelled particles from the nearby dimers. This process repeats, yielding an oscillatory state. Since the density of active particles in this oscillatory state is saturated with unexpectedly large fluctuations even for $\rho<\rho_c$, the power-law behavior of Eq. (1) was not observed.

In the second case, in order to avoid oscillatory states, only one of the particles in a dimer is allowed to hop and the remaining particle is considered to become inactive. We believe that this rule interprets the physical situation better than the first update rule. In computer programming, the system is usually swept from left ($i=1$) to right ($i=L$) in order to search for active particles and to determine the sites to which the active particles are to hop in the next time step. If one always chooses the left particle in a dimer to hop and the right particle to become inactive, one would get interesting results. If one proceeds with dynamics simultaneously while sweeping the system, one may possibly perform this way because the left particle in a dimer is checked earlier. However, since there is no particular reason to always choose the left particle to be active, this "asymmetric" parallel update rule is erroneous. The density of active particles at $\rho=\rho_c$ was found to decay algebraically over time, with the power of $\theta=\frac{1}{2}$. The value of θ, which is twice as large as that of the random sequential update, indicates that the convergence behavior of ρ_a is much faster and that 10^6 steps are already sufficient with regard to measuring the order parameter. The order parameter was calculated for the selected values of ρ and the results are plotted in Fig. 4(a). In the main plot, the data display linear behavior on an ordinary scale, implying that $\beta=1$, which is consistent with the analytical result [24]. The inset is the double-logarithmic plot of the same data with the slope being precisely 1. Both off-critical scaling and finite-size scaling hold for $\nu_\parallel=2$ and $z=1$, as shown in Figs. 4(b) and 4(c). We however found that the density of active particles at ρ_c decayed rapidly and the system always fell into the absorbing state. Therefore, the scaling form in Eq. (7) cannot be applied and, accordingly, $\nu_\perp$ cannot be obtained. We summarize the results as

$$\theta=\frac{1}{2},\quad \beta=1,\quad \nu_\parallel=2,\quad z=1.$$

It should be noted that this set of compact results is different from that of the random sequential update.

We understand the dynamics of the CLG model in one dimension as follows. As the dynamics proceeds, the system rapidly goes into the state in which most active particles consist of dimers. In the sequential update, dimers are repeatedly split and reconstituted before system goes into absorbing states, while in the asymmetric parallel update, since the left particle always hops, dimers are split much more quickly. The difference is more apparent when we consider the perturbed situation from one of the absorbing states. Suppose that an inactive particle near the center of the system is moved to one of its nearest-neighbor site so that the perturbed particle and the particle on the neighboring site become active. In the random sequential update, the pair of active particles follow the random walks, yielding $z=2$.

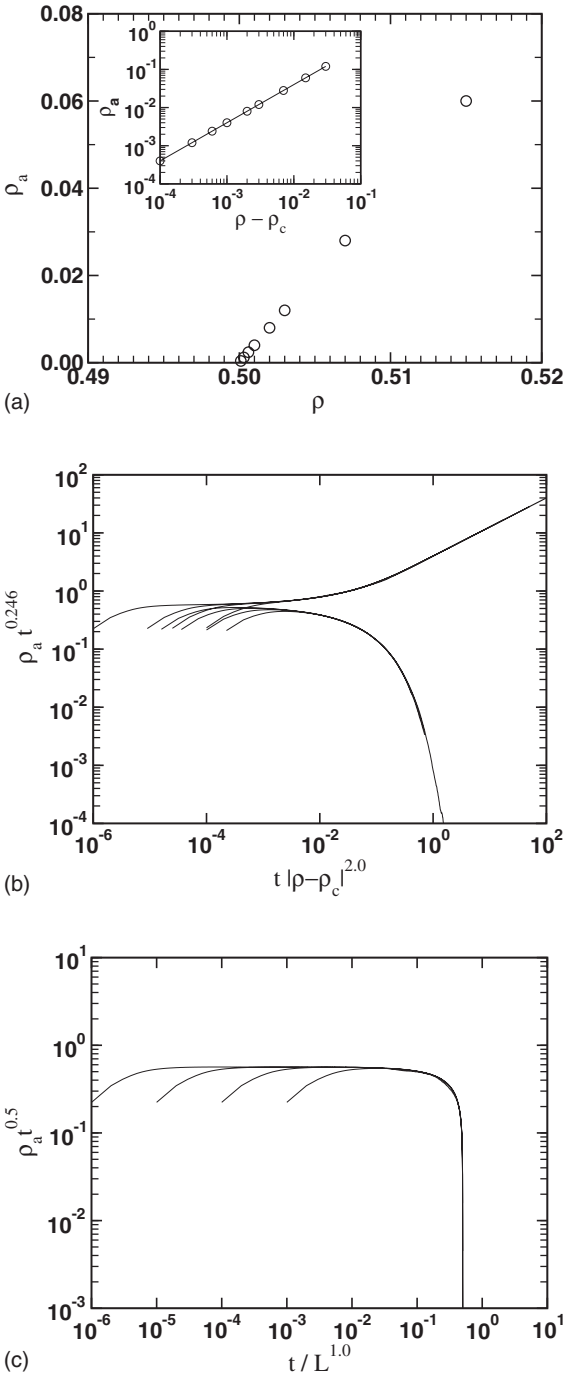

FIG. 4. Monte Carlo data obtained using the second parallel update rule. (a) is the order parameter of the system plotted against the density of particles. Plotted in the inset are the same data on a double-logarithmic scale as a function of the distance from the criticality. (b) is the off-critical scaling function $\rho_a t^{\theta}$ plotted against the scaled variable $t|\rho-\rho_c|^{\nu_\parallel}$, for $\rho=0.485$, 0.49, 0.494, and 0.496 (lower data) and $\rho=0.501$, 0.503, 0.505, and 0.51 (upper data) for the size of system $L=10^6$. (c) is the finite-size scaling function plotted against the scaled variable t/L^z for $L=10^3$, 10^4, 10^5, and 10^6.

However, in the asymmetric parallel update, the pair of active particles follow the self-avoiding walk, yielding $z=1$.

TABLE I. Local configurations of lattice sites and hopping probabilities of active particles. Active sites are marked by • and inactive sites by ◦, and P_A, P_B, P_C, and P_D represent the probabilities of hopping marked, respectively, as A, B, C, and D on the configurations.

Lattices	P_A	P_B	P_C	P_D
A B • • •→◦←• • •	1/2	1/2		
A B ◦ ◦←• •→◦ ◦	1/2	1/2		
A B C • • •→◦←• •→◦ ◦	3/4	1/4	1/2	
A B C D ◦ ◦←• •→◦←• •→◦ ◦	1/2	3/8	3/8	1/2
A B C ◦ ◦←• •→◦←• • •	1/2	1/4	3/4	

Finally, we present the results which are consistent with those of the sequential update. When active particles in a dimer attempt to hop, it is reasonable to select a hopping particle randomly and the remaining particle is set to be inactive. This will evenly distribute the hopping probability; however, the situation is not simple when two active particles attempt to hop to the same site. In Table I, we summarize all possible cases, with the relevant hopping probabilities of the active particles. The probabilities in the table are symmetric; i.e., when sweeping the system from right to left, the probability distribution is identical to that obtained by sweeping in the other direction.

In the first step, we search for all active particles by scanning the system from left to right and, in the second, we determine the sites to which the active particles hop in the next time step. When more than one active particle attempts to hop to the same site, we choose a hopping particle following the probabilities given in Table I. With this rule, we found that ρ_a exhibits a power-law behavior at ρ_c, with the critical index $\theta \simeq 0.246$. The power-law behavior of an order parameter, the off-critical scaling, and the finite-size scaling plot that were obtained using the critical indices $\beta=0.984$, $\nu_\parallel=4.0$, and $\nu_\perp=1.95$ are similar to the plots in Figs. 1 and 2. Therefore, we do not present them here. As a result, the estimates are consistent with those for the sequential update.

In summary, we studied the absorbing phase transition of the CLG model in one dimension using both the sequential and parallel updates. We found that the order parameter exponent β was consistent with the known analytical results for both cases. We also estimated the other exponents such as θ, $\nu_\parallel$, and $\nu_\perp$. We found that the results were consistent for the sequential and parallel updates only when the hopping probabilities of active particles were assigned to be symmetric. The estimates were consistent with the results of the model by Park *et al.*, supporting that the CLG model and their model exhibit the same critical behavior in one dimension. Our conclusion is contrasted with the claim made by Park *et al.* in their work.

We also found that the off-critical scaling and the finite-size scaling are satisfied with our estimates; however, the known scaling relation $z=\nu_\parallel/\nu_\perp$ in absorbing phase transi-

tion is violated. We surmise that such a failure is caused by the finite-size effect which is always present in the CLG model.

This work was supported by Korea Science and Engineering Foundation Grant No. R01-2004-000-10148-0 and also by the BK21 Project.

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